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Signal Processing 84 (2004) 805–811
SIGNAL PROCESSING
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Adaptive approximation of feedback rank filters for continuous signals

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Received 3 February 2003

Abstract

Rank-based nonlinear filtering techniques are steadily gaining in popularity due to their robustness. However, the analog implementation of these techniques meets with considerable conceptual and practical difficulties. Here we describe an adaptive approximation for a rank filter of a continuous signal expressed in terms of a system of differential equations easily implementable in an analog circuit. The design is based on consideration of the finite precision of physical measurements, which leads to simple and efficient implementation of many traditionally digital analysis tools. We also illustrate the performance of the adaptive approximation filter in comparison with the respective 'exact' rank filter in a boxcar moving window.

Keywords: Adaptive analog rank filters; Analog signal processing; Nonlinear filters

1. Introduction

The main advantage of analog signal processing with respect to digital processing is simple implementation and efficient handling of nonlinearities. However, there are many signal processing tasks for which digital algorithms are well known, but corresponding analog operations are hard to reproduce. One example which falls within this category is related to the use of signal processing techniques based on *order statistics*,[1] such as implementing median and other order statistic filtering [16]. Order statistic filters are gaining wider recognition for their ability to provide more robust estimators of signal properties. For example, the *median* value of a set of measurements usually represents the general trend in a signal better than the *mean* value, since the latter is more sensitive to outliers. However, while analog implementation of the mean is trivial, median estimators are much harder to implement in analog form [5–7], since, traditionally, their determination involves the operation of sorting or ordering a set of measurements. Indeed, there is no conceptual difficulty in sorting a set of discrete measurements, but it is much less obvious how to perform similar operations for continuous signals [3,4].

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[1] See, for example, [1,15] for the definitions and theory of order statistics.

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doi:10.1016/j.sigpro.2004.01.001

As pointed out by some authors [14], the major problem in analog rank processing is the lack of an appropriate differential equation for 'analog sorting'. There have been many attempts to implement such sorting and build continuous-time rank filters. Examples of these efforts include optical rank filters [12], analog sorting networks [13,14], and analog rank selectors based on minimization of a nonlinear objective function [17]. However, the term 'analog' is often perceived as only 'continuous-time', and these efforts fall short of considering the *threshold continuity*, which is necessary for a truly analog representation of differential sorting operators. Even though the recent work by Ferreira [4] extensively discusses threshold distributions, these distributions are only piecewise-continuous and thus do not allow straightforward introduction of differential operations with respect to threshold.

Recently, we have proposed a new approach to constructing analog devices for performing traditionally digital signal processing tasks [9–11]. This approach is based on the consideration of the finite precision of real measurements, with the resulting modification of the definitions of various signal properties and underlying mathematical equations. Since analog systems are implemented using physical components, the mathematical description of such systems must take into account their limited precision and inertial characteristics. Therefore, the output of an analog device typically represents a weighted average over a nonzero time and threshold intervals. Realization of this fact enables us to rewrite many problems of signal analysis in the form readily addressed by methods of differential calculus, which are suitable for analog implementation, rather than by the algebraic or logic operations of the digital approach. In [9], we have outlined the general principles of this approach and suggested several applications. In the present article we apply these principles to develop a simple and accurate approximation for a rank filter of a continuous signal in a boxcar moving window.

2. Continuous discriminators and probes

Consider a simple measurement process whereby a signal $x(t)$ is compared to a threshold value D. The ideal measuring device would return '0' or '1' depending on whether $x(t)$ is larger or smaller than D. The output of such a device is represented by the Heaviside unit step function $\theta[D-x(t)]$, which is discontinuous at zero. However, the finite precision of real measurements inevitably introduces uncertainty in the output whenever $x(t) \approx D$. To describe this property of a real measuring device, we represent its output by a *continuous* function $\mathscr{F}_{\Delta D}[D-x(t)]$, where the *width parameter* ΔD characterizes the threshold interval over which the function changes from '0' to '1' and, therefore, reflects the measurement precision level. We call $\mathscr{F}_{\Delta D}(D)$ the threshold *step* response of a *continuous discriminator*. Because of the continuity of this function, its derivative $f_{\Delta D}(D) = \mathrm{d}\mathscr{F}_{\Delta D}/\mathrm{d}D$ exists everywhere, and we call it the discriminator's threshold *impulse* response, or a *probe* [9,11]. This *threshold continuity* of the output of a discriminator is the key to a truly analog representation of such a measurement. Examples of step and impulse responses of a continuous discriminator are shown in Fig. 1. We further assume, for simplicity, that the probe is a unimodal even function, that is, $f_{\Delta D}(D)$ has only a single maximum and $f_{\Delta D}(D) = f_{\Delta D}(-D)$.

In practice, many different circuits can serve as discriminators, since any continuous monotonic function with constant unequal horizontal asymptotes will produce the desired response under appropriate scaling and reflection. It may be simpler to implement a discriminator described by an odd function $\tilde{\mathscr{F}}_{\Delta D}$ which relates to the response $\mathscr{F}_{\Delta D}$ as

$$\tilde{\mathscr{F}}_{\Delta D} = A(2\mathscr{F}_{\Delta D} - 1), \tag{1}$$

where A is an arbitrary (nonzero) constant. For example, the voltage–current characteristic of a subthreshold transconductance amplifier [8,17] can be described by the hyperbolic tangent function, $\tilde{\mathscr{F}}_{\Delta D} = A\tanh(D/\Delta D)$, and thus such an amplifier can serve as a continuous discriminator. For specificity, this response function is used in the numerical example of this article. A practical implementation of the respective probe $f_{\Delta D}$ of the

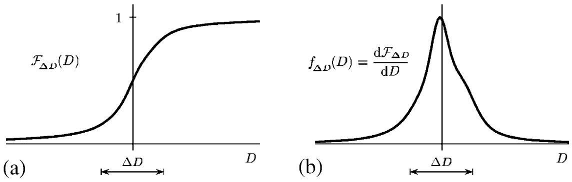

Fig. 1. Representative step (a) and impulse (b) responses of a continuous discriminator.

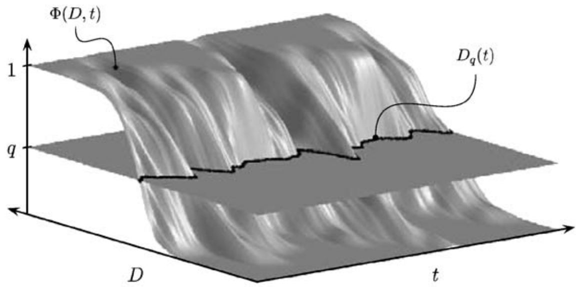

Fig. 2. Defining output $D_q(t)$ of a rank filter as a level curve of the distribution function $\Phi(D,t)$.

discriminator $\mathscr{F}_{\Delta D}$ can be conveniently accomplished as a finite difference

$$f_{\Delta D}(D) \approx \frac{\delta\tilde{\mathscr{F}}_{\Delta D}(D)}{4A\delta D} = \frac{1}{4A\delta D}[\tilde{\mathscr{F}}_{\Delta D}(D+\delta D) - \tilde{\mathscr{F}}_{\Delta D}(D-\delta D)], \tag{2}$$

where δD is a relatively small fraction of ΔD.

3. Analog rank filters

Consider the measuring process in which the difference between the *threshold variable D* and the scalar signal $x(t)$ is passed through a discriminator $\mathscr{F}_{\Delta D}$, followed by a linear time averaging filter with a continuous impulse response $w(t)$. The output of this system can be written as

$$\Phi(D,t) = w(t) * \mathscr{F}_{\Delta D}[D - x(t)], \tag{3}$$

where the asterisk denotes convolution. The physical interpretation of the function $\Phi(D,t)$ is the (time dependent) *cumulative distribution function* of the signal $x(t)$ in the moving time window $w(t)$ [9]. In the limit of high resolution (small ΔD), Eq. (3) describes the 'ideal' distribution [4]. Notice that $\Phi(D,t)$ is viewed as a function of *two* variables, *threshold* D and time t, and is continuous in both variables.

The output of a *quantile* filter of order q in the moving time window $w(t)$ is then given by the function $D_q(t)$ defined implicitly as

$$\Phi[D_q(t),t] = q, \quad 0 < q < 1. \tag{4}$$

Viewing the function $\Phi(D,t)$ as a surface in the three-dimensional space (t,D,Φ), we immediately have a geometric interpretation of $D_q(t)$ as that of a level (or contour) curve obtained from the intersection of the surface $\Phi = \Phi(D,t)$ with the plane $\Phi = q$, as shown in Fig. 2. Based on this geometric interpretation, one can develop various explicit as well as feedback representations for analog rank filters, including such generalizations as L filters and α-trimmed mean filters [9]. We will further focus on a particular feedback representation of the basic rank filter for signal amplitudes.

4. Rank filter in RC window

When the time averaging filter in Eq. (3) is an RC integrator (RC$=\tau$), a differential equation for the output $D_q(t)$ of a rank filter takes an especially simple form and can be written as

$$\frac{dD_q}{dt} = \frac{A(2q-1) - \tilde{\mathscr{F}}_{\Delta D}[D_q(t) - x(t)]}{2A\tau h_\tau(s) * f_{\Delta D}[D_q(t) - x(s)]|_{s=t}}, \tag{5}$$

where $h_\tau(t) = \theta(t)\exp(-t/\tau - \ln\tau)$.[2] The solution of this equation is ensured to rapidly converge to $D_q(t)$ of the chosen quantile order q regardless of the initial condition [9]. Note also that the continuity of the discriminator is essential for the right-hand side of Eq. (5) to be well behaved.

The main obstacle to a straightforward analog implementation of the filter given by Eq. (5) is that the convolution integral in the denominator of the right-hand side needs to be re-evaluated (updated) for each new value of D_q. If we wish to implement an analog rank filter in a simple feedback circuit, then we should replace the right-hand side of Eq. (5) by an approximation which can be easily evaluated by such a circuit. Of course, one can employ a great variety of such approximations [2, for example], whose suitability will depend on a particular goal. A very simple approximation becomes available in the limit of sufficiently small τ, since then we can replace $h_\tau(s) * f_{\Delta D}[D_q(t) - x(s)]|_{s=t}$ by $h_\tau(t) * f_{\Delta D}[D_q(t) - x(t)]$ in Eq. (5). As was shown in [9], this simple approximation can still be used for an arbitrary time window $w(t)$, if we represent $w(t)$ as a weighted sum of many RC integrators with small τ. We now provide an example of using this technique for approximating an output of a rank filter in a moving boxcar time window.

5. Adaptive approximation of a feedback rank filter in a boxcar time window

A rank filter in a boxcar moving time window $B_T(t) = [\theta(t) - \theta(t-T)]/T$ is of a particular interest, since it is the most commonly used window in digital rank filters. The output D_q of an analog rank filter in this window is implicitly defined as $B_T(t) * \mathscr{F}_{\Delta D}[D_q - x(t)] = q$. To construct an approximation for this filter suitable for implementation in an analog feedback circuit, we first approximate the boxcar window $B_T(t)$ by the following moving window $w_N(t)$:[3]

$$w_N(t) = \frac{1}{N}\sum_{k=0}^{N-1} h_\tau(t - 2k\tau), \tag{6}$$

where $\tau = T/(2N)$. The first moments of the weighting functions $w_N(t)$ and $B_T(t)$ are identical, and the ratio of their respective second moments is $\sqrt{1 + 2/N^2} \approx 1 + 1/N^2$. The other moments of the time window $w_N(t)$ also converge rapidly, as N increases, to the respective moments of $B_T(t)$, which justifies the approximation of Eq. (6).

Now, the output of a rank filter in such a window can be approximated as discussed at the end of Section 4, namely as [9][4]

$$\frac{\mathrm{d}D_q}{\mathrm{d}t} \approx \frac{AN(2q-1) - \sum_{k=0}^{N-1} \tilde{\mathscr{F}}_{\Delta D}[D_q(t) - x(t - 2k\tau)]}{2A\tau h_\tau(t) * \sum_{k=0}^{N-1} f_{\Delta D}[D_q(t) - x(t - 2k\tau)]}, \tag{7}$$

where $\tau = T/(2N)$. Note that the accuracy of this approximation is contingent on the requirement that $\Delta D > |h_\tau(t) * \dot{x}(t)|\tau$. This means that, if we wish to have a simple analog circuit and keep N relatively small, we must choose ΔD sufficiently large for the approximation to remain accurate. On the other hand, we would like to maintain high resolution of the acquisition system, that is, to keep ΔD small.

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[2] In more explicit notation, the convolution integral in the denominator of Eq. (5) can be written as

$$h_\tau(s) * f_{\Delta D}[D_q(t) - x(s)]|_{s=t} = \frac{1}{\tau}\int_{-\infty}^{t} \mathrm{d}s \exp\left(\frac{s-t}{\tau}\right) f_{\Delta D}[D_q(t) - x(s)].$$

[3] Since a moving time window is always a part of a convolution integral, the approximation is understood in the sense that $B_T(t)*g(t) \approx w_N(t) * g(t)$, where $g(t)$ is a smooth function

[4] An explicit expression for the convolution integral $h_\tau(t) * f_{\Delta D}[D_q(t) - x(t - 2k\tau)]$ is

$$h_\tau(t) * f_{\Delta D}[D_q(t) - x(t - 2k\tau)] = \frac{1}{\tau}\int_{-\infty}^{t} \mathrm{d}s \exp\left(\frac{s-t}{\tau}\right) f_{\Delta D}[D_q(s) - x(s - 2k\tau)].$$

In order to reconcile these conflicting requirements, we propose to use an *adaptive* approximation, which reduces the resolution only when necessary. This can be achieved, for example, by using Eq. (2) and rewriting the threshold derivative of $h_\tau(t) * \tilde{\mathscr{F}}_{\Delta D}[D_q - x(t)]$ as

$$h_\tau(t) * f_{\Delta D}[D_q - x(t)] \approx \frac{h_\tau(t) * \{\tilde{\mathscr{F}}_{\Delta D}[D_{q+} - x(t)] - \tilde{\mathscr{F}}_{\Delta D}[D_{q-} - x(t)]\}}{4A(D_{q+} - D_{q-})}, \tag{8}$$

where $D_{q\pm}$ is the output of a rank filter of the quantile order $q \pm \delta q$, $\delta q \ll q$. In essence, the approximation of Eq. (8) amounts to decreasing the resolution of the acquisition system only when the amplitude distribution of the signal broadens, while otherwise retaining high resolution.

Combining Eqs. (6)–(8), we arrive at the following representation of an adaptive approximation to a feedback rank filter in a boxcar time window of width T:

$$\begin{aligned}
\dot{D}_q(t) &= \frac{AN(2q-1) - \sum_{k=0}^{N-1} \tilde{\mathscr{F}}_{\Delta D}[D_q(t) - x(t-2k\tau)]}{h_\tau(t) * \delta\tilde{\mathscr{F}}_{\Delta D}(t)} \, \frac{\delta D_q(t)}{\tau},\\
\dot{D}_{q+}(t) &= \frac{AN(2q-1+2\delta q) - \sum_{k=0}^{N-1} \tilde{\mathscr{F}}_{\Delta D}[D_{q+}(t) - x(t-2k\tau)]}{h_\tau(t) * \delta\tilde{\mathscr{F}}_{\Delta D}(t)} \, \frac{\delta D_q(t)}{\tau},\\
\dot{D}_{q-}(t) &= \frac{AN(2q-1-2\delta q) - \sum_{k=0}^{N-1} \tilde{\mathscr{F}}_{\Delta D}[D_{q-}(t) - x(t-2k\tau)]}{h_\tau(t) * \delta\tilde{\mathscr{F}}_{\Delta D}(t)} \, \frac{\delta D_q(t)}{\tau},
\end{aligned} \tag{9}$$

where $\delta D_q(t) = D_{q+}(t) - D_{q-}(t)$ and

$$\delta\tilde{\mathscr{F}}_{\Delta D}(t) = \sum_{k=0}^{N-1} \{\tilde{\mathscr{F}}_{\Delta D}[D_{q+}(t) - x(t-2k\tau)] - \tilde{\mathscr{F}}_{\Delta D}[D_{q-}(t) - x(t-2k\tau)]\}. \tag{10}$$

This approximation preserves its validity for high-resolution discriminators (small ΔD), and its output converges, as N increases, to the output of the 'exact' rank filter in the boxcar time window $B_T(t)$. The accuracy of this approximation is best described in terms of the error in the quantile q. That is, the output $D_q(t)$ can be viewed as bounded by the outputs of the 'exact' rank filter for different quantiles $q \pm \Delta q$. When ΔD and δq in Eq. (9) are small, the error range Δq is of order $1/N$.

Fig. 3 compares the performance of the analog rank filter given by Eq. (9) to that of the 'exact' quantile filter in a boxcar moving window of width T. In this example, the quantile interval δq is chosen as $\delta q = 10^{-2}$ (1%). The continuous input signal $x(t)$ (shown by the solid dark gray line) is emulated as high resolution time series (2×10^3 points per interval T). The 'exact' outputs of a boxcar window rank filter are shown by the dashed lines, and their deviations within the $\pm\Delta q$ intervals are shown by the gray bands. The respective outputs of the approximation given by Eq. (9) are shown by the solid black lines. The width parameter ΔD of the discriminators, the width T of the boxcar time window, the quantile order q, and the number N of exponential kernels in the approximation are indicated in the figure.

The (instantaneous) accuracy of the approximation given by Eq. (9) decreases when the input signal $x(t)$ undergoes a large (in terms of the resolution parameter ΔD) monotonic change over a time interval of order τ. The main effect of such a 'sudden jump' in the input signal is to delay the output $D_q(t)$ relative to the output of the respective 'exact' filter. This delay is shown as Δt in the lower left portion of the upper panel, where the input signal is a square pulse. This timing error Δt is inversely proportional to the number N of the kernels in the approximation. The accuracy of the approximation can also be described in terms of the *amplitude* error. As can be seen in Fig. 3, the residual oscillations of the outputs of the analog filter occur within the $q \pm 1/(2N)$ interval around the respective outputs of the 'exact' filter (that is, within the width of the gray bands in the figure).

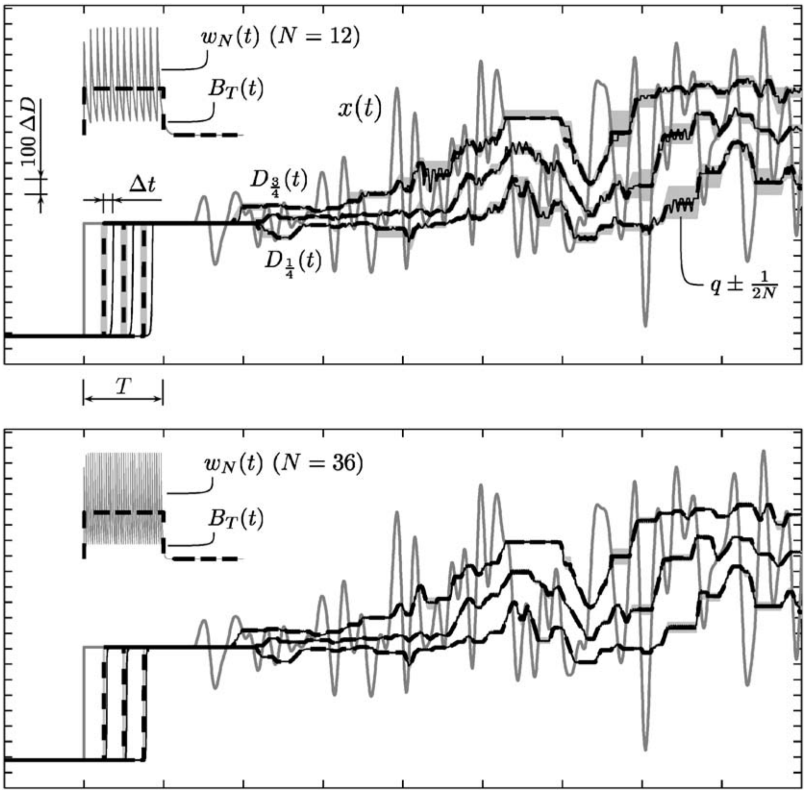

Fig. 3. Illustration of the performance of the analog rank filter given by Eq. (9) by comparing its quartile outputs ($q=\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$, solid black lines) with the respective outputs of the 'exact' order statistic filter in a rectangular moving window B_T of width T (dashed lines).

6. Conclusion

This article describes an adaptive approximation of a real-time rank filter, suitable for implementation in an analog feedback circuit. Both the input and output of this filter are continuous signals. The width of the moving window and the quantile order are continuous parameters as well, and such continuity can be utilized in various analog control systems. The adaptivity of the approximation allows us to maintain a high resolution of the discriminators regardless of the properties of the input signal, which enables the usage of this filter for nonstationary signals.

Finally, let us point out that the equation describing this filter is also suitable for numerical computations, especially when the number of data points within the moving window is large. A simple forward Euler method is fully adequate for integrating this equation, and the numerical convolution with an RC impulse response function requires remembering only one previous value. Thus a numerical algorithm based on this equation has the advantages of both high speed and low memory requirements.

Acknowledgements

We express our sincere appreciation to Jane H. MacGibbon and Denis V. Popel, both of Baker University, and Thomas P. Armstrong of the University of Kansas and Fundamental Technologies, LLC, for their valuable suggestions and critical comments.

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